1971 von Kármán Lecture

Trends in the Field of Automatic Control in the Last Two Decades

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THIS lecture is given in memory of Theodore von Kármán. There are two reason which make me happy to be this year's lecturer. I have known Dr. von Kármán personally. I met him the first time at a conference in England in 1934. The other reason is the admiration I have had for his work since I started working in Göttingen's Aerodynamics Research Institute in 1929, where von Kármán spent time working with L. Prandtl and where he was well remembered.

In 1940 von Kármán delivered the fifteenth J. W. Gibbs lecture at a meeting of the American Mathematical Society, titled "The engineer grapples with nonlinear problems." He explained that in many fields of engineering nonlinearities could no longer be neglected, and that a deeper mathematical treatment of some problems had led to particularly interesting results.

Since in my lecture nonlinear control theory and its growth will play a role, it seems to me that this topic is fit to remind us of von Kármán and his far-reaching ideas.

Speaking before an audience of aeronautical engineers, I cannot forget those times when "control" meant stabilization of an airplane with the help of control surfaces at the tail or ailerons at the wing. The pilot used them to fly a desired path and to counteract gusts, or adverse winds.

There came a step forward when the pilot's task was eased by building the "automatic pilot" to which the human pilot would essentially give commands.

My talk today shall not be restricted to this specific task, but consider "automatic control" in general, built to relieve the human being from any dull and repetitive operation, may be to supervise a chemical process or may be to determine the path of a rocket, a missile and finally to let satellites fulfill their task without manual interference.

After World War II automatic controls have spread into many regions of technology. The consequence was that barely an engineer in any field could escape coming in contact with them. Some mechanisms and regulators were a long time in use, (for instance, the centrifugal regulator of steam engines), but complicated automatic control systems were still little used.

Beginning with the 1950s, the mathematicians' interest awoke and this was the start of a tremendously fast development surpassing the linear control theory which was presented in a number of books written for engineers with relatively minor mathematical background.

Let us go back to the airplane. One understood the basic behavior of the dynamic system, and the controls tended to keep the plane on its desired path and in its desired attitude by removing the undesired deviations which could be described by linear differential equations. Linear control elements were mostly used, however, at the same time the airplane was a wonderful example to show that linearization was not always suitable for describing the behavior of the system, and controls designed on that assumption could not be satisfactory. Maybe nonlinear controls had to be introduced. Among others, Kochenburger and MacColl[1,2] had attacked the problem of designing stable nonlinear control systems.

I remember a conference at which two wind tunnel designers were enthusiastic about their success in designing a wind tunnel with special flow characteristics. After linear controls did not yield the desired airstream quality, they finally used Kochenburger's paper and, with the help of the describing function they designed a suitable control of the propulsion system.

I could not agree only with their conclusions that Kochenburger's rule had solved all nonlinear control problems.

The describing function allows us to handle certain nonlinear controls in the same manner as linear ones by way of approximation. J. G. Truxal's book on Control System Synthesis[3] contains many historically interesting details. The essential assumptions for its use is: the system contains only one nonlinear element; the output of this element depends only on the present value and the past history of the input; if the input is sinusoidal only the fundamental output of the nonlinear element influences the system.

One particular nonlinear type of control used was the *discontinuous control*; often designed with the help of the describing function, or by trial and error method. In this case, each of the control inputs has only two or three settings. In wartime, the controls were employed for steering missiles. Such controls are simple from the design standpoint, they are rugged and inexpensive. These qualities are appreciated in controlling objects which

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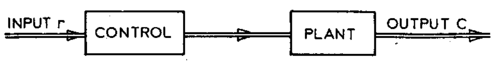
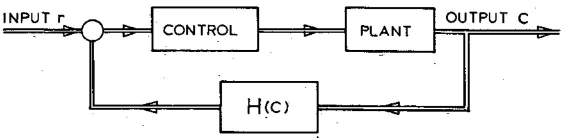
Dr. Irmgard Flügge-Lotz attended the Technische Hochschule in Hannover, Germany, from 1923 to 1927, receiving the degree of Dipl.-Ing. in 1927. She became an assistant in Applied Mechanics and worked for the degree of Dr. Ing., which she received in 1929. She was with the Aerodynamics Research Institute in Göttingen from 1929 to 1938, advancing from research scientist to Head of the Department of Theoretical Aerodynamics. From 1939 to 1945, she was a consultant on aerodynamics and the dynamics of flight for the DVL (German Aeronautical Research Establishment). She served as Chief of a Research Group in Theoretical Aerodynamics at ONERA in Paris during 1947 and 1948. Since 1949, Dr. Flügge-Lotz has been at Stanford University, where she was appointed Professor of Aeronautical Engineering and Engineering Mechanics in 1960 and was awarded the title of Professor Emerita in 1968. Dr. Flügge-Lotz is the author of two books, *Discontinuous Automatic Control* (1953) and *Discontinuous and Optimal Control* (1968), and of about fifty technical papers. She is a Fellow of the AIAA, a Member of Sigma Xi, and recipient of the 1970 Achievement Award from the Society of Women Engineers. Her fields of specialization are theoretical aerodynamics and automatic control.



a)

a) Open-loop (= programed) control



b)

b) Feedback control

Fig. 1 Block diagrams.

get destroyed in fulfilling their mission. Their only, but important deficiency was that their theory was not widely known*; partially, also not fully developed. There was, however, the great advantage that the over-all nonlinear discontinuous control was piecewise linear. This allowed for instance a relatively simple search for periodic motions which had to be avoided if a missile was to follow its desired path and experienced outside disturbances.

Let us assume that everybody is familiar with the concepts of forward and of feedback control. The two sketches in Fig. 1 will be a quick reminder to realize the enormous advantage of feedback. The input may be considered as the command (programed control). The boxes can be filled with mathematical symbols indicating their content, depending largely on the mathematical description of the plant. The double lines are a convenient means to indicate that there may be more than one input, output, etc.

Note that Laplace transforms are not introduced in Fig. 1. This would restrict the figure to describing linear systems only. About 1950 much was known about control of linear systems. The goal was to design good controls. The quality of the controls was mostly judged by the response of the system to sinusoidal inputs or to step inputs; time delays to step inputs and phase-shift of the responses to sinusoidal inputs were observed and improved if desired by the choice of the free parameters in the control element. This design was based on the idea that various inputs could be represented by superposition of sinusoidal motions. A general discussion of other performance measures seemed not yet to be urgent; however, it was evident that as soon as one should leave the realm of linear systems, one had to face the necessity of defining "performance" in a more general manner.

In many cases this could be easily done for certain technical tasks. For instance, if the ouput should be close to the command input ($\vec{r}-\vec{c}=\vec{e}<\varepsilon$) in spite of unavoidable disturbances the plant might undergo, one could measure the performance during the operation time (t_1-t_0) by the integral

$$J=\int_{t_0}^{t_1}|e|^2\,dt \tag{1}$$

and consider the best control that one which made J a minimum. This leads inevitably to a variational problem. Only variational calculus could deliver the minimum if it existed. If for the control one would choose in advance, an analytical function with free

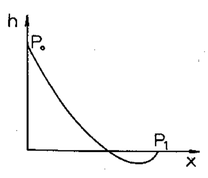


Fig. 2 Impossible trajectory for a landing airplane.

* For details see Ref. 4.

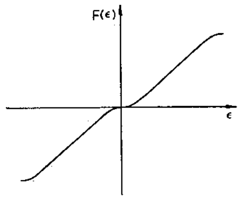


Fig. 3 Nonlinear control function.

constant parameters, one could not expect to get the true minimum, but only a low value of J depending on the chosen control function. The variational method, however, can deliver the exact form of the control function which leads to a minimum.

There were tasks, particularly for flying objects, where reaching a goal (going from one point in the vector space to another one) should be done in minimum time. This problem attracted many researchers. It is particularly interesting because it shows immediately how carefully such variational problems have to be formulated. If the x axis in Fig. 2 presents the Earth surface, a landing of an airplane starting at altitude h_0, etc.... and aiming at $h=0$ in P_1 could not follow a path as sketched,† because it would have to go into the ground and appear unscathed again, which is technically not possible. Performance criteria may, therefore, have to be accompanied by strict limits on the state variables.

In the first decade, 1950–1960, one investigated linear systems thoroughly. For one input-one output systems, they could be described by an nth order ordinary differential equation or by a system of n first-order differential equations. Upon using vector notation, $\mathbf{x}$ an $(n\times 1)$ vector, $\mathbf{u}$ an $(m\times 1)$ control vector, A an $(n\times n)$ matrix, G an $(n\times m)$ matrix and B an $(n\times n)$ matrix, the system is described by

$$\dot{\vec{x}}=A\vec{x}+G\vec{u};\qquad \vec{z}=B\vec{x} \tag{2}$$

where z_i are the observable variables. A, B, and G are determined by the dynamic or electronic system under consideration. $\vec{u}$ is determined by the desired performance criterion.

Since systems to be controlled may be stable or unstable without control, one of the tasks of the control is to obtain stability in the entire working region of the process.

For investigating the stability of an obtained controlled linear system there are the well-known Hurwitz or Routh criteria available.

R. Kalman introduced the concepts of *controllability* (i.e. the ability to transform an arbitrary initial state to an arbitrary final state in finite time) and *observability*,‡ of linear systems. The ample use of matrix theory made the handling and description easier, particularly since electronic computers allowed to obtain matrices fast.

Parallel work was going on to study nonlinear systems with weak nonlinearities (for instance hardening or weakening springs). A number of nonlinear dynamic systems have been described by N. Minorsky in his work, which in addition about 1950 brought to the attention of the Western World the work done in Russia. His first book[7] presented more material about the stability of dynamical systems than about control systems in particular; however, his work has certainly influenced the building of controls at that time.

Particular attention found a type of control system studied by the Russian A. I. Lurje.[8] One of his examples may be cited, the automatic control of an airplane (simplified; longitudinal motion only).

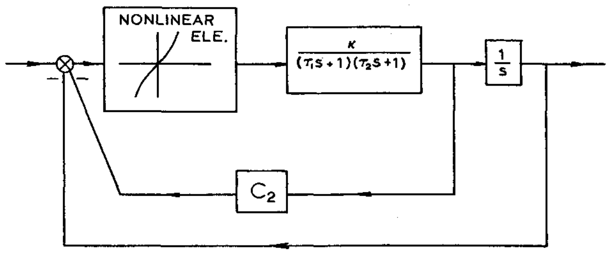
$$T\ddot{\phi}+\dot{\phi}=-\kappa\delta,\qquad \dot{\delta}=F(\varepsilon)$$

with

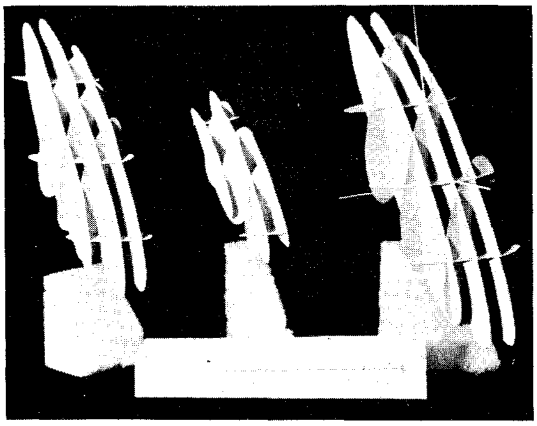
$$\varepsilon=\kappa\phi-\kappa_{\dot{\phi}}\dot{\phi}+\kappa_\delta\delta \tag{3}$$

A nonlinear function F is given in Fig. 3. The stability of such systems can no longer be investigated with Hurwitz or Routh criteria. Liapunov's direct method was considered the ideal tool for this purpose. On the suggestion of Prof. S. Lefschetz, the Princeton University Press published Liapunov's original papers

† This problem was treated in Ref. 5.
‡ See Chap. 11 of Ref. 6.



a) Block diagram



b) Stability region computed with Zubov's method for polynoms of 2nd, 4th, and 6th order

Fig. 4 Nonlinear third-order system

in French in 1947. Liapunov defined a function of the state variables which can be used to determine whether a system is stable or not without finding the solution of the differential equations which describe the system. Many scientists felt this function to be the most important tool of the future. However, as soon as one had to work with Liapunov's theorems, one found that their very generality made finding a Liapunov function a new hard task.§ Let us look at the so called second theorem: Given a dynamic system

$$\text{with} \qquad \begin{aligned} \dot{x}_m &= F_m(x_1, x_2, x_3, \ldots, x_n) \\ x^T &= x_1, x_2, x_3, \ldots, x_n \end{aligned} \tag{4}$$

If one can choose a positive (or negative) definite Liapunov function V so that its derivative with respect to time along the trajectories of the system, $\dot{V} = W(x_1, x_2, \ldots, x_n)$ is also definite and has the opposite sign, the system is asymptotically stable.

If the system variables do not cover the entire vector space, the Liapunov's function may give the bounding surface of the stability region. There is not the place to go into all details; however, it is important to state that mostly only for low-order systems Liapunov functions can be found rather easily. It should be mentioned that the Liapunov function is in general not equal to the energy content of the system, an idea which is often very attractive if one does not go beyond second-order systems.

Zubov[10] has given a differential equation for finding Liapunov functions V. For a second-order system, for instance,

$$\dot{x}_1 = f_1(x_1, x_2) \qquad \text{and} \qquad \dot{x}_2 = f_2(x_1, x_2) \tag{5}$$

V must satisfy the differential equation

$$dV/dt = (\partial V/\partial x_1)f_1 + (\partial V/\partial x_2)f_2 = -\phi(x_1, x_2)[1 - V] \tag{6}$$

Zubov could show that a good choice of the function ϕ only accelerates the finding of a good Liapunov function by iteration, but that any functions ϕ will lead to the best Liapunov function for the problem under consideration, if the iteration converges.

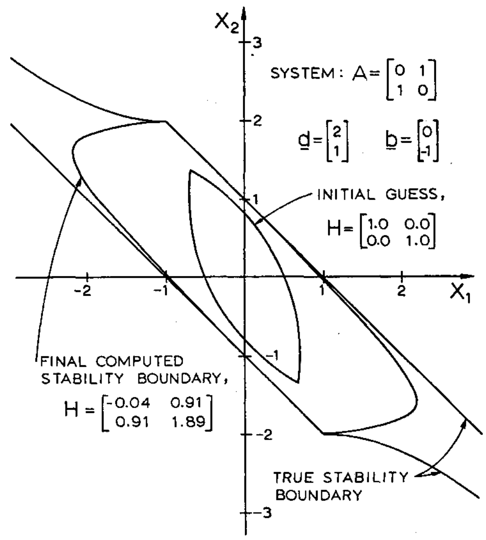
§ Many readers will enjoy reading the Monograph.[9]

A number of years have let us see suggestions for new types of Liapunov functions. Finally one had to conclude that it was a very attractive mathematical problem, but that even with big computers at hand it was a tedious job to get results for high-order problems (see Ingwerson, Rodden, Weissenberger).[11-13]

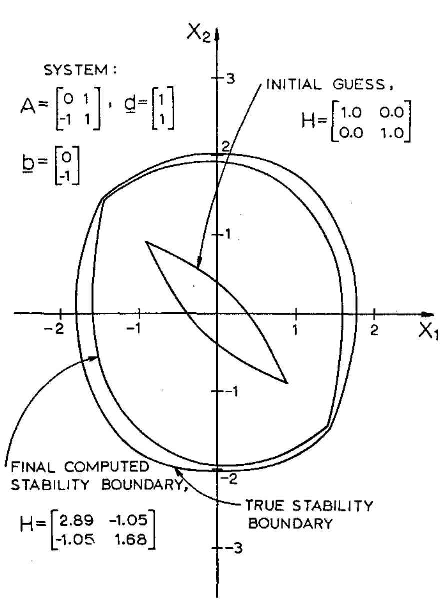
To give an idea of the difficulties, Fig. 4a shows the block diagram of a third-order system, and Fig. 4b shows the stability boundaries for a third-order system which Rodden investigated. Zubov suggests as a V function a series of mth-order polynomials

$$V(x_1, x_2) = \sum_{k=2}^{k=\infty} V_k(x_1, x_2) \tag{7}$$

The lowest approximation with $k_{max} = 2$ gives a stability boundary enclosing a volume which is bigger than the next approximation with $k_{max} = 4$. And $k_{max} = 6$ gives a stability region which is larger than that obtained with $k_{max} = 2$. That means the convergence of the iteration process is often not monotonous.

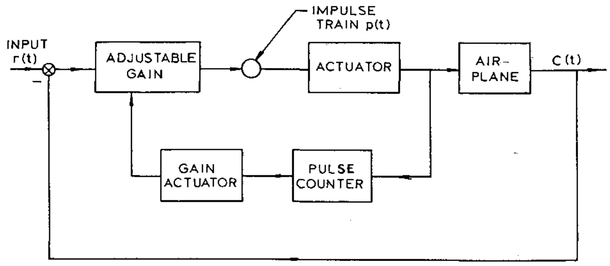


a) Liapunov function of the type Eq. (8) is used. True stability region is an infinite strip

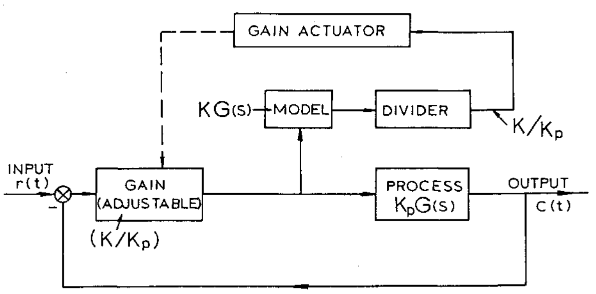


b) Same Liapunov function type for a system with finite stability region

Fig. 5 Stability boundaries for systems.
$\dot{x} = Ax + b\,\mathrm{sgn}\,[d'x]$



a) Simplified Sperry autopilot



b) Example of an adaptive system which uses model of the plant

Fig. 6 Two adaptive systems.

Weissenberger used for a second-order discontinuous system the Liapunov function¶

$$V = \tfrac{1}{2}\vec{x}^T H \vec{x} + \vec{d}^T \vec{x}\,\mathrm{sgn}(\vec{d}^T \vec{x}) \tag{8}$$

The choice of the matrix H leads to a sequence of boundaries of the stability region. That H which gives the maximum stability region gives the best Liapunov function. The H values are marked in Fig. 5a, b. It is interesting to see that H = identity matrix is not a good choice.

Thus far it was assumed that the system to be controlled was well known; that means coefficients of the derivatives in the differential equations describing the system were either constant or known functions of time and eventually of the state variables (nonlinear case). In the second half of the 1950's much attention was devoted to the control of systems whose dynamics changes during operation time. This could be caused for instance if a flying object entered a new environment. Certainly the controls had to take these facts in account and much thought was devoted to studying and designing "adaptive controls." For some times engineers were searching for an exact defintion of "adaptivity." Therefore, one finds older articles within about 1960 which call system adaptive which may be good control systems, but do not deserve this name. The situation is well described in "Adaptive Control Systems"[14] edited by Eli Mishkin and Ludwig Braun Jr. J. S. Truxal wrote the introduction chapter "The Concept of Adaptive Control." The figures show the situation well. In Fig. 6a** the impulse train allows the measurement of the "over-all system damping," in the second figure a comparison of a "model" of the desired process to the actually changed process is used to adjust the gain of the controller. Even if process and model are linear, the over-all controlled process is nonlinear. The design of the model is a special task for which different methods are available. However, all will depend on the input type one used for expressing the intput-output relation of the uncontrolled not analytically known process.

While these investigations (discontinuous systems, slightly nonlinear, nonlinear and adaptive systems) were going on, the optimization problem raised its head, one may say, and became the dominant topic in the second decade. The performance of a control design was mentioned earlier. There were two ways to approach the goal; one, the Dynamic Programing[15] suggested by R. Bellman in the midfifties and, second, Pontryagin's Maximum Principle presented to the Western World in 1958 in Edinburgh.

The first is a method which demands big computing equipment which the 1960's provided amply. With all respect for its inventor, it seems to me always a somewhat brutal method of attacking a problem. One subdivides the state space in the following manner. One marks possible state points at a certain time t_κ. Then one takes one particular point which lies on an optimum trajectory and computes the cost (performance dependent) to go to all points in the state point set at time $t_{\kappa+1} = \Delta t$. The chosen path (part of the state trajectory) is the connection from the starting point to a particular point in the set of statepoints at time $t_{\kappa+1}$ which gives the lowest cost and will in continuing the procedure finally lead to the final state point at the time t_f.

Bellman was fully aware of the amount of storage space needed for handling problems of high dimensionality, he spoke of the "curse of dimensionality." However, if one overcomes this difficulty, then one can solve even optimization problems which other methods would not be able to handle. If the performances are linear functions of the state variables, one can use linear programing, which requires much less storage space.[16]

The second method was published in several Russian journal articles starting 1956. (Later Pontryagin and his associates wrote a book, that was translated into English.[17]) Pontryagin's method is attractive, in spite of the fact that one gets the control function as function of time. That means the feedback design has still to be done, which requires to express the control function as function of the state variables.

Pontryagin's Maximum Principle in its original form gives a necessary condition for computing the control function for the following problem:

system equations $\dot{x}_i = f_i(x_j, u_k, t)$

$\mathbf{x}$ is the state vector $(n \times 1)$

$\mathbf{u}$ is the control vector $(m \times 1)$ (9)

t is the time

Initial condition $\vec{x}(0)$ and final condition $\vec{x}(t_f)$ are given. Final time may or may not be given. The performance criterion is

$$F = \int_0^{t_f} g(x, u, t)\,dt \to \text{minimum} \tag{10}$$

The control vector has mostly upper and lower limits, the state variables may be restricted to a limited region of the state space.

It seems evident that variational calculus can be used for finding the unknown control function; however, one has to take in account that, if the optimal solution is assumed to be known, one can only study variations which satisfy the constraints and limit conditions.

A very simple example†† may illustrate the problem:

$$\text{system} \quad \dot{x} = u_2 + 1 \tag{11}$$

Performance criterion is

$$\int_0^{t_1} (u_1 + 1)\,dy \to \text{minimum} \tag{12}$$

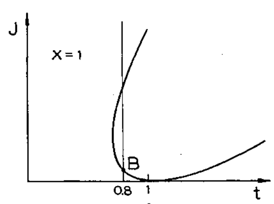


Fig. 7 Performance values for fixed state variable x and varying mission time.

¶ This form cannot be used when the trajectory slides along the switching surface. For details, see Weissenberger's paper.

** Which represents a simplified Sperry auto-pilot.

†† See Halkin's example, pp. 162–164 in Flügge-Lotz, I., *Discontinuous and Optimal Control,* McGraw-Hill, New York, 1968.

with the condition that the constant control components u_1, and u_2 satisfy

$$u_1^2+u_2^2 \leqq 1$$

$$\text{Initial State } x(0) = 0; \quad \text{Final State } x(0.8) = 1 \qquad (13)$$

Figure 7 shows the possible value of J. The minimum is in B and is not zero, which is the lowest value of J for $x = 1$, but which contradicts the conditions on u_1 and u_2.

The difficulty of Pontryagin's method lies in the fact that it requires the solution of a two-point boundary problem. The differential equations are all of first-order and for an nth order system consist of two groups, those for the n state variables and those for the n so-called adjoint variables. The $2n$ boundary conditions are given for the n state variables. That means, if one wants to work with numerical integration, one has to guess the initial adjoint variables which would deliver the correct state variables at the final time.

A number of algorithms have been developed to improve the possibilities of solving two-point boundary value problems. Methods of "steepest descent" have been employed often; they use mostly successive linearization if they are iterative procedures.

If the performance criterion has a more complicated form, e.g.,

$$J = g_0(\vec{x}_f, t_f) + \int_{t_0}^{t_f} g_1(\vec{x}, \vec{u}, t)\,dt \rightarrow \min \qquad (14)$$

Pontryagin's method still works. However, if for instance one should have a condition that during the entire mission time ($t_0 \rightarrow t_f$) the acceleration of the state point in the state space never is allowed to pass beyond a certain value, then one encounters difficulties, and has to use Bellman's Dynamic Programing.[18]

Also limited state space can produce quite a trouble, when working with Pontryagin's method. The statepoint trajectory may lie partially on the boundary; it may also experience a kind of reflection (corner-condition) when it hits the boundary. In the latter case an additional difficulty turns up. One does not know in advance how many "corner-conditions" will have to be satisfied, because one does not know in advance how often the boundary surface will be hit.

The description of solving optimal control problems is by far not complete; but in a paper of this type it only is possible to indicate essential trends of the solution possibilities. In fact, in the second decade 1960–1970, the publishing market was flooded with journal articles and books to a degree which seems to have never occurred with another topic. Particularly obvious became the tendency to make optimal control a pure mathematical topic, described in the mathematical language which does so often not care for the need and comfort of the engineering readers.

One aspect of optimal controls has really not gained enough attention. It has been pointed out what an enormous role the chosen performance criterion plays. However, it is not always clear which of several performance criteria is the most important one for a specific design. Not even the designer of a technical object may say that, because the controls are different for different criteria, but the design of a specific control may be so cumbersome that one asks what should happen if another one would have been chosen. Certainly, it is possible to pose an optimization problem that considers a performance criterion which is a weighted contribution of several. The simple way is probably the one suggested by Flügge-Lotz,[19] evaluating the controls for, assume, 3 performance criteria J_1, J_2, J_3 as functions of the mission time ($t_f - t_0$). After this is done, the control making J_1 a minimum is used and the costs $J_2{}^*$ and $J_3{}^*$ are computed. They are by no means identical to J_2 and J_3; however, their determination requires only integrations. Then one takes the control for the performance criterion J_2 and computes the costs $J_1{}^{**}$ and $J_3{}^{**}$. Finally, the optimal control for the criterion J_3 is chosen, and the costs $J_1{}^{***}$ and $J_2{}^{***}$ are computed by mere integration. Now certain interesting diagrams are ready for the designer and by comparing diagrams‡‡ in-frames indicated in

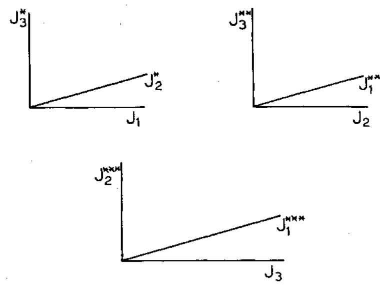


Fig. 8 Frames for comparing different performance criteria.

Fig. 8 he can make a decision what he would consider a good compromise.

Thus far, it has been assumed that state variables and controls are known or can be observed or computed at any desired time t with sufficient accuracy. However, the so called "noise," that means random disturbances which influence many processes has been neglected. More often than not random noise occurs and its influence has to be accounted for.§§ In such cases, for instance satellite control, in the entire problem variables have to be replaced by their expected or probable values. The optimization idea per se is the same, only deterministic values are replaced by estimated values.

Tasks on Earth, but more so investigation in the universe will bring with them the necessity to consider the influence of transport delays.

Systems with transport lag, for instance,

$$\dot{x}_m = f_m(x_1-\theta_1, x_2-\theta_2, \ldots, u_1-v_1, u_2-v_2 \ldots) \qquad (15)$$

have been facing the engineer for a long time. It is known that uncontrolled system which are described by differential-difference equations can be quite cumbersome. A system of nth order experiencing delays will show the features of a system of infinite order. The stability investigation is quite difficult and therefore has attracted many mathematicians. N. H. Choksy published a bibliography on "Time-Lag Systems" in IRE transactions, Vol. AC-5, No. 1, 1960. Oğuztörelli[21] investigated the time-lag problem further; however, a real intensive presentation of theory with examples is still missing. R. Bate[22] treated the problem by starting with a differential-integral equation of the type

$$\dot{x}(t) = \int_{-\infty}^{t} f[x(t'), u(t'), t', t]\,dt' \qquad (16)$$

This type could be related to

$$\dot{y}(t) = \int_{-\infty}^{t} [F(t,t')y(t') + D(t,t')u(t')]\,dt' \qquad (17)$$

for linear plants. For special choices of F and D, the last equation can lead to

$$\dot{y}(t) = \sum_{i=0}^{p} A_i(t)y(t-\theta_i{}^*) + \sum_{j=0}^{q} B_j(t)u(t-v_j{}^*) \qquad (18)$$

It follows that under specified realistic condition the solution of the last Eq. (18) can be obtained by solving the preceding differential-integral equation with computer help. For details, see Bate's paper and his interesting examples.

Conclusion

In this article an attempt has been made to show the essential events in the controls field in the last two decades. It was unavoidable that a number of interesting ideas, procedures and facts have not been mentioned (for instance, the singular solution in optimal control, the use of the "penalty" function for designing trajectories which do not traverse the boundary of the allotted state-space, etc.). However, this article's basic idea was not to be a complete survey, but to give the engineer (who is not expected to be a controls specialist) some aspects which have attracted great attention. The reason for the strong development is that automation,

‡‡ For details see Ref. 19.

§§ See, for intance, Ref. 20.

big computing equipment (digital essentially, but also the enormous improvement of analog computers and the development of hybrid¶¶ computers) and the desire to investigate the universe appeared in this period.

The great interest of mathematicians in making use of matrix calculus and a deeper understanding of systems of ordinary differential equations has sometimes pushed in directions that are interesting, but technically not of long-time strong influence. One can see that the Liapunov theorems have had their peak influence time, as has had the minimum time solutions of technical problems.

The cool view of the open-minded designer will determine which of the new control ideas will be used, and what will be admired mostly as a theoretical jewel in spite of its usefulness in some very specific problems.

¶¶ Combination of analog and digital computers.

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